Theory Qual

Fall 2016

Please answer all four questions below.

1. For a graph G and a subset of nodes S, let $G \setminus S$ denote the subgraph obtained when S and all of its incident edges are removed from G. Define the following Node-Deletion Problem:

   Input: A graph $G = (V, E)$ where every vertex has degree at most 3, and integer k.

   Output: "Yes", if there is a subset $S \subset V$ with $|S| \leq k$ such that $G \setminus S$ is bipartite.

   "No", otherwise.

   Prove that the Node-Deletion Problem is NP-complete. You may use the fact that MAX-CUT is NP-complete for graphs with maximum degree 3.

2. You are given a sorted circular linked list containing n integers, where every element has a "next" pointer to the next larger element. (The largest element's "next" pointer points to the smallest element.) You are asked to determine whether a given target element belongs to the list. The only way you can access an element of the list is to follow the next pointer from a previously accessed element, or via the function RAND that returns a random element of the list.

   Develop a randomized algorithm for finding the target that accesses at most $O(\sqrt{n})$ elements in expectation. If the target is present in the list, your algorithm should return a pointer to it, and if it is not, your algorithm should return "Error".

3. Given a graph $G = (V, E)$, and a subset T of vertices, a T-join is a set of edges $E' \subseteq E$ such that in the subgraph $G' = (V, E')$, the degree of every node in T is odd and the degree of every node in $V \setminus T$ is even. Let w be a weight function on edges, $w : E \to \mathbb{R}^+$, and let $w(E') = \sum_{e \in E'} w_e$ denote the weight of E'. Develop a polynomial time algorithm for finding the minimum weight T-join given G, T, and w.

4. [Corrected.] Let $b, n \geq 2$. This problem concerns finite state machines that take as input a string x of base-b digits and compute the value of $x \bmod n$. The machines are deterministic, and the output is determined by the final state the machine is in.

   (a) Show that if the input x is read from left to right, such a machine can be constructed using $O(n)$ states.

   (b) Show that if the input is read from right to left, any such machine must have $\Omega(n^2)$ states (for infinitely many n).

   (Hint: You may use the fact that if n is prime and b is a generator for $\mathbb{Z}_n^*$, the period of the sequence b^i is $n - 1$.)