

**Spring 2014 Qualifier Exam:
OPTIMIZATION**

February 3, 2014

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered. *Do not write your name on any answer book.*
3. Return all answer books in the folder provided. Additional answer books are available if needed.

SPECIFIC INSTRUCTIONS:

Answer 4 out of 5 questions.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. You have a set J of jobs that must be scheduled within a set of time periods $\mathcal{T} = \{1, \dots, T\}$. Each job has an integer processing time $p_j > 0$. When being processed, jobs use capacity from a set I of machines. In particular, when job $j \in J$ is being processed it requires $a_{ij} > 0$ units of the capacity of machine $i \in I$. At any point in time, the total available capacity of machine $i \in I$ is denoted by $b_i > 0$. Jobs can be processed simultaneously, provided the machine capacity constraints are not exceeded, but jobs cannot be interrupted (i.e., once a job starts it will be in process for p_j consecutive time periods). In the following questions, you should use (at least) the following “time-indexed” decision variables to determine the start-time of the jobs.

- $x_{jt} = 1$ if job $j \in J$ starts at time $t \in \{1, 2, \dots, T - p_j\}$, $x_{jt} = 0$ otherwise.
- (a) Write a linear integer programming formulation to minimize the sum of start times of the jobs.
- (b) Suppose now each job $j \in J$ has an earliest start-time $r_j \geq 1$ and a latest completion time $D_j \leq T$. One possible way to model these restrictions is with the constraints:

$$r_j \leq \sum_{t \in \mathcal{T}} tx_{jt} \leq D_j - p_j, \quad j \in J. \quad (1)$$

However, these constraints would lead to a weak linear programming relaxation. Provide a *different* way to model the start-time and completion time restrictions, and argue why your model is preferred.

- (c) Now suppose that you find it is not feasible to schedule all the jobs so that they are completed by their latest completion time D_j . Thus, you wish to relax this constraint, and instead penalize lateness in the objective. Specifically, if job $j \in J$ completes at time $t > D_j$, then this will be penalized by $(D_j - t)^2$. Modify your model to replace the objective with the objective of minimizing the sum of penalties. (Your model must remain an integer *linear* program.)
- (d) Now consider a pair of jobs j and k that have a precedence relationship: job k cannot begin processing until job j completes its processing. This can be modeled with the constraint

$$\sum_{t \in \mathcal{T}} tx_{jt} + p_j \leq \sum_{t \in \mathcal{T}} tx_{kt}. \quad (2)$$

However, this constraint again leads to a weak linear programming relaxation. Provide an alternative model for this restriction that would lead to a better relaxation. (Hint: the formulation should involve inequalities that only have coefficients 0 or 1 on the decision variables). You *do not* have to provide a proof that the LP relaxation of your formulation is better than (2).

2. Consider the parametric linear program $PLP(\theta)$:

$$\begin{aligned}
 z(\theta) = \min \quad & 4x_1 + 2x_2 \quad \quad + x_4 \\
 \text{s.t.} \quad & x_1 \quad \quad - x_3 + x_4 \quad \quad = \theta \\
 & x_1 + x_2 \quad \quad \quad \quad = \theta \\
 & x_1 \quad \quad - x_3 \quad \quad - x_5 \quad \quad = 1 \\
 & \quad \quad x_2 \quad \quad \quad \quad + x_6 = 1 \\
 & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
 \end{aligned}$$

- (a) Write the dual linear program of $PLP(\theta)$.
- (b) Determine $z(\theta) \forall \theta \in \mathbb{R}$.
- (c) Is $z(\theta)$ a convex function, a concave function, or neither?
- (d) Is your answer from part (c) generally true? That is, for the general parametric linear program, where the right-hand side vector b is treated as a parameter, is the value function $z(b)$,

$$z(b) \stackrel{\text{def}}{=} \min_{x \geq 0} \{c^\top x \mid Ax = b\},$$

a convex function of b , a concave function of b , or neither? Provide a proof of your claim.

- (e) Let $x_4^*(\theta)$ be an optimal value for the decision variable x_4^* as a function of the parameter θ in $PLP(\theta)$. (For $PLP(\theta)$, $x_4^*(\theta)$ is a single-valued mapping (a function)). Determine $x_4^*(\theta) \forall \theta \in \mathbb{R}$.
- (f) Is $x_4^*(\theta)$ a convex function, a concave function, or neither?

3. Let $S = \{x^1, x^2, \dots, x^t\}$ be a finite set of points in \mathbf{R}^n , and let $P = \{x \in \mathbf{R}^n : Ax \leq b\}$ be a non-empty polytope where A is an $m \times n$ matrix and $b \in \mathbf{R}^m$. Let $\hat{x} \in \mathbf{R}^n$ be given. Formulate a linear program that determines whether or not $\hat{x} \in \text{conv}(S \cup P)$, and if not identifies a valid inequality for $\text{conv}(S \cup P)$ that is violated by \hat{x} . State how the linear program answers the question and prove that it provides a correct answer (this requires proving something in its two possible statements: $\hat{x} \in \text{conv}(S \cup P)$ and $\hat{x} \notin \text{conv}(S \cup P)$).

4. Assume that X is a bounded polyhedron. Let $x^0 \in X$ be given and for $k \geq 0$ let \bar{x}^k be defined as an extreme point of X satisfying

$$\bar{x}^k \in \arg \min_{x \in X} \nabla f(x^k)^T (x - x^k),$$

and suppose that x^{k+1} is a stationary point of the optimization problem

$$\min_{x \in X^k} f(x),$$

where X^k is the convex hull of x^0 and the extreme points $\bar{x}^0, \dots, \bar{x}^k$.

- (a) Write down the definition of X^k explicitly.
- (b) Define the notion of a stationary point for $\min_{x \in X} f(x)$.
- (c) Under what conditions is \bar{x}^k well defined? Identify a (known) algorithm that can determine \bar{x}^k .
- (d) Show that there exists a finite integer k such that the above method finds a stationary point of f over X .

5. (a) Consider the following semidefinite program, in standard form:

$$\min_{X \in S\mathbb{R}^{2 \times 2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bullet X \text{ subject to } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \bullet X = 1, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \bullet X = 0, X \succeq 0.$$

Write down the dual of this problem, and find the complete primal and dual solution sets, together with the optimal objective value for both problems.

- (b) Consider the following semidefinite program, in standard form:

$$\min_{X \in S\mathbb{R}^{2 \times 2}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \bullet X \text{ subject to } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \bullet X = 0, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bullet X = 2, X \succeq 0.$$

Write down the dual of this problem, and find the complete primal and dual solution sets, together with the optimal objective value for both problems.

- (c) Give sufficient conditions on a primal-dual pair of semidefinite programs that guarantees that the solutions sets of both problems are *nonempty* and *bounded* and have equal objective value. Are these conditions satisfied by the problems in parts (a) and (b)?