Theory Qual

Fall 2018

Please answer all four questions below.

1. Show that the following problem is NP-complete: Given positive integers $n, n_1, \ldots, n_k$ in binary representation, decide whether n can be written as a product $\prod_{i=1}^{k} n_i^{e_i}$ where the e_i's are nonnegative integers.

2. You are playing a video game with n levels. The goal is to pass all of the levels and reach the finish line without accumulating too many "demerit points". In each level ℓ, you have k actions available. Action i in level ℓ brings you $d_{\ell;i}$ demerit points and has a probability $p_{\ell;i}$ of succeeding. If the action succeeds, you pass this level and move on to level $\ell + 1$. However, if it fails, you are sent back to level 1 and must solve all of the levels again. Given the quantities $d_{\ell;i}$ and $p_{\ell;i}$for each pair (ℓ, i), design an algorithm that runs in time $\mathrm{poly}(n, k)$ and finds a strategy for playing the game that minimizes the expected number of demerit points accumulated before the game ends.

3. The feedback vertex set problem is as follows: We are given an undirected graph $G = (V, E)$, and our goal is to find the smallest subset of vertices whose removal makes the graph acyclic. The "dual" to this problem asks to find the largest set of vertex-disjoint cycles in the graph G.

   (a) Prove that the size of any feasible solution to the dual problem provides a lower bound on the size of any feasible solution to the feedback vertex set problem.

   (b) Given a feasible solution to the dual problem, consider the set of vertices in the union of all cycles in the dual solution. Prove that if the dual solution is *maximal*,[1] then this set of vertices is a feedback vertex set. What approximation to the feedback vertex set problem does this imply?

   (c) Consider the following algorithm for constructing a primal and dual solution in tandem: while the graph contains a cycle, pick one such cycle C; Include the cycle C in the dual solution; Remove a subset S of the vertices in the cycle C from the graph and include this subset in the primal solution; Iterate.

   Figure out how to choose the cycle C and the subset S in each iteration so as to obtain a good approximation for the feedback vertex set problem. Aim for a bound of $O(\log n)$ on the approximation ratio, where n is the number of vertices in G.

   Hint: You may want to use the fact that every n-vertex graph with minimum degree 3 contains a cycle of length at most $2 \log n$.

4. For $k = 0, 1, 2, \ldots$, let $\Delta_k^e := \mathrm{DTIME}(2^{O(n)})^{\Sigma_k^p}$. This is called the linear-exponential hierarchy. Show that for some $k \in \{0, 1, 2, \ldots\}$, Δ_k^e contains a language that has maximum circuit complexity at every input length. Pick k as small as possible.

[1] That is, no more cycles can be added to the solution without violating vertex-disjointness.