Please answer all four questions below.

1. For a graph \( G \) and a subset of nodes \( S \), let \( G \setminus S \) denote the subgraph obtained when \( S \) and all of its incident edges are removed from \( G \). Define the following Node-Deletion Problem:

   **Input:** A graph \( G = (V, E) \) where every vertex has degree at most 3, and integer \( k \).

   **Output:** “Yes”, if there is a subset \( S \subseteq V \) with \(|S| \leq k\) such that \( G \setminus S \) is bipartite.

   “No”, otherwise.

   Prove that the Node-Deletion Problem is NP-complete. You may use the fact that MAX-CUT is NP-complete for graphs with maximum degree 3.

2. You are given a sorted circular linked list containing \( n \) integers, where every element has a “next” pointer to the next larger element. (The largest element’s “next” pointer points to the smallest element.) You are asked to determine whether a given target element belongs to the list. The only way you can access an element of the list is to follow the next pointer from a previously accessed element, or via the function RAND that returns a random element of the list.

   Develop a randomized algorithm for finding the target that accesses at most \( O(\sqrt{n}) \) elements in expectation. If the target is present in the list, your algorithm should return a pointer to it, and if it is not, your algorithm should return “Error”.

3. Given a graph \( G = (V, E) \), and a subset \( T \) of vertices, a \( T \)-join is a set of edges \( E' \subseteq E \) such that in the subgraph \( G' = (V, E') \), the degree of every node in \( T \) is odd and the degree of every node in \( V \setminus T \) is even. Let \( w \) be a weight function on edges, \( w : E \rightarrow \mathbb{R}^+ \), and let \( w(E') = \sum_{e \in E'} w_e \) denote the weight of \( E' \). Develop a polynomial time algorithm for finding the minimum weight \( T \)-join given \( G, T \), and \( w \).

4. [Corrected.] Let \( b, n \geq 2 \). This problem concerns finite state machines that take as input a string \( x \) of base-\( b \) digits and compute the value of \( x \mod n \). The machines are deterministic, and the output is determined by the final state the machine is in.

   (a) Show that if the input \( x \) is read from left to right, such a machine can be constructed using \( O(n) \) states.

   (b) Show that if the input is read from right to left, any such machine must have \( \Omega(n^2) \) states (for infinitely many \( n \)).

   (Hint: You may use the fact that if \( n \) is prime and \( b \) is a generator for \( \mathbb{Z}_n^* \), the period of the sequence \( b^i \) is \( n - 1 \)).