1. In the Balanced Path problem, we are given a directed graph $G$ with special nodes $s$ and $t$, and a partition of the edges of the graph into $k$ groups. The goal is to determine whether there is a path from $s$ to $t$ that contains at most one edge of any one group. Prove that the Balanced Path problem is NP-complete.

2. Suppose that you possess a black-box for solving instances of the Knapsack problem exactly. You want to use this black-box for solving the following Multiple Knapsack problem: you are given $n$ items of values $v_i$ and weights $w_i$ respectively, as well as $k$ knapsacks, each of capacity 1. Your goal is to find a collection of disjoint subsets of items $S_1, \ldots, S_k$, such that each subset fits into a single knapsack, that is, for all $j \in [k]$, where $[k] = \{1, 2, \ldots, k\}$, $\sum_{i \in S_j} w_i \leq 1$, and the total value $\sum_{i \in \bigcup_{j \in [k]} S_j} v_i$ is maximized.

Consider the following greedy algorithm: (1) Use the single-Knapsack black-box to find the optimal subset of items for the first knapsack. (2) Recurse for the remaining knapsacks over the remaining items.

(a) Prove that the greedy algorithm does not always return the optimal solution.

(b) Prove that the greedy algorithm obtains a factor of $3/2$ approximation for $k = 2$.

3. A branching program (BP) on Boolean variables $\{x_1, \ldots, x_n\}$ is defined as follows. It is a directed graph where vertices are partitioned into layers $L_i$ ($1 \leq i \leq m$), for some $m$. For each $1 \leq i < m$, there exists $1 \leq j \leq n$, such that $L_i$ is labeled by $x_j$, and every node in $L_i$ has two edges to nodes in $L_{i+1}$ labeled 0 and 1 respectively. One node in $L_1$ is the starting node $a$, and one node in $L_m$ is the accepting node $z$. Any assignment $\sigma : \{x_1, \ldots, x_n\} \rightarrow \{0, 1\}$ defines a unique path from $a$ to some node in $L_m$ by following the edges consistent with $\sigma$. The BP is said to compute a Boolean function $f(x_1, \ldots, x_n)$ if: $\sigma$ defines a path from $a$ to $z$ iff $f(\sigma(x_1), \ldots, \sigma(x_n)) = 1$.

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in (\mathbb{Z}_2)^{3 \times 3}$ be two matrices over the finite field $\mathbb{Z}_2$. Note that $ABAB = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Build a BP where each node corresponds to a matrix of the form $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$, where $(a, b, c) \in \{0, 1\}^3$, to compute the following sequence of functions:

- $f_1 = x_1 \land x_2$, and $g_1 = x_1 \lor x_2$.
- $f_2 = (x_1 \lor x_2) \land (x_3 \lor x_4)$, and $g_2 = (x_1 \land x_2) \lor (x_3 \land x_4)$. 

• $f_k = g_{k-1}(x_1, \ldots, x_{2^{k-1}}) \land g_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})$, and $g_k = f_{k-1}(x_1, \ldots, x_{2^{k-1}}) \lor f_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})$.

Your BP should have polynomial length in the number of variables $2^k$ for $f_k$ and $g_k$.

4. Consider the following statement: For every $L \in \text{NL}$ there exists a constant $c$ such that for every positive integer $d$, $L$ can be decided by circuits of depth $d$ and size $2^{nc/d}$ for almost all input lengths $n$. Show that the statement:

(a) fails for circuits of bounded fan-in, and
(b) holds for circuits of unbounded fan-in.

Good Luck!!