CS559 – Lecture 13
Curves

These are course notes (not used as slides)
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Final version (after class 10/18)

Shape Modeling

- Creating Mathematical Descriptions of Shape
- Why?
  - Drawing, Sample, Analyze
- Why is this hard
  - Shapes can be arbitrary and complex – hard to describe
  - Conflicting goals
    - Concise
    - Intuitive
    - Expressive
    - Analyzable
    - …

What is a Shape

- Mathematical definition is elusive
- Set of Points
  - Potentially (usually) infinite
- "Lives" in some bigger space (e.g. 2D or 3D)
- Many ways to describe sets
  - Set inclusion test (implicit representation)
  - Procedural for generating elements of the set
  - Explicit mapping from a known set

Some kinds of Shapes

- Curves
  - 1D Objects, like what you draw with a pen
- Surfaces / Areas
  - 2D Objects – the insides of 2D things
  - Bounded by a Curve
- Solids / Volumes
  - 3D Objects – the insides of things that take up volume
  - Different definition: set with the same dimension as the embedded space (an area of 2D)

Curves

- Intuitively, something you can draw with a pen
  - Not filled areas
  - Mathematical oddity: space filling curves
    - Requires infinite lengths, …
- Almost every point has 2 "neighbors"
- Locally equivalent to a line

Defining Curves

- Two different mathematical definitions
  1. The continuous image of some interval
  2. A continuous map from a one-dimensional space to an n-dimensional space
- Both definitions imply a mapping
  - From a line segment (which is a curve)
- #1 is a set of points, #2 is the mapping

Describing Curves

- Some curves have names
  - Line, line segment, ellipse, parabola, circular arc
- Some set of parameters to specify
  - Radius of an arc, endpoints of a line, …
- Other curves do not have distinct names
  - Need a *Free Form* representation

Curve Represenations

- Implicit
  - Function to test set membership
  - F(x,y) = 0
- Explicit / Parametric
  - Y = f(x)
  - (x,y) = f(t) – where t is a free parameter
    - Need to define a range for the parameter
- Procedural
  - Some other process for generating points in the set
- By definition, a curve has at least 1 parametric representation

Parameterizations

- For any curve (set of points) there may be many mappings from a segment of the reals
- Consider: line from 0,0 -> 1,1
  - (x,y) = (t,t) $t \in [0,1]$
  - (x,y) = (.5t, .5t) $t \in [0,2]$
  - (x,y) = (t^2,t^2) $t \in [0,1]$
- Many ways to represent a curve

Free Parameters

- Not really a property of the curve
  - Many different parameterizations
- Think of it as time in the pen analogy
  - Parameterization says "where is pen at time T"
  - Many different ways to trace out the same curve have different timings
- Can "reparameterize" a curve
  - Same curve, different parameterization
  - Add a function f(t) -> f(g(t)) $g \in$ R->R

Some nice Parameterizations

- Unit Parameterization
  - Parameter goes from 0 to 1
  - No need to remember what the range is!
- Arc-Length Parameterization
  - Constant magnitude of 1^{st} derivative
  - Constant rate of free parameter change = constant velocity
  - Arc-length parameterizations are tricky

How do we define functions?

- Simple shapes: easy
- Complex shapes, divide and conquer
  - Break into small pieces, each an easy piece
  - Approximate if needed
    - Add more pieces to get better approximations
  - Need to make sure pieces connect
- Typically, pick simple, uniform pieces
  - Line segments, polynomials, …

Parametric Values for Compound Curves

- Could reparameterize however we want
- One parameter space for all pieces
- Switching at various points
- KNOTS are the switching points
  - (0, .5, 1) in the case below

$$f(u) = \begin{cases} f_1(2*u) & \text{if } 0 \leq u < \frac{1}{2} \\ f_2(2*u-1) & \text{if } \frac{1}{2} \leq u < 1 \end{cases}$$

Connecting Pieces

- Only concerned about the knots
  - Assume the pieces are smooth
- Connection & Smoothness
  - Connection is a type of smoothness
- Derivative continuity
  - 0^{th} derivative = position
  - 1^{st} derivative = direction
  - 2^{nd} derivative = curvature

Types of Continuity

- C(n) continuity
  - Derivatives up to (and including N) match
  - May have less meaning since parameterizations don't mean anything
- G(n) continuity
  - C(0)
  - Higher derivatives may differ by a scale factor
- How smooth?
  - C(2) = smooth in graphics
  - Higher continuity in design (boat hulls, …)