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Revisions

The revisions were made according to the referees' suggestions as mentioned in the bold and italic statements below.

Reviewer #1: You may have missed the recent paper

A classification of transitive ovoids, spreads and m-systems of polar spaces, Forum Mathematicum 21(2009), 181-216

which classifies primitive ovoids in all classical polar spaces provided they do not have cardinality a prime number. I recommend that you add it to your list of references, and refer to it in Section 1 (after discussing Kleidman's result) and/or in Section 5, when referring to your own result about primitive ovoids from [15].

Added to the list of references. Referred to it in Section 1 and Section 5.

Typos : voids for ovoids (page 13, line 9), bipartitie for bipartite (page 6, line 21); there is no Tits ovoid over the field of order 27 (page 13, line 28), but they should have occurred two sentences earlier in the list of known ovoids. Page 13, line 27, you need two definite articles added : For $q=25$ the only... For $q=27$ the only...

Typos were corrected.

Reviewer 2

The authors introduce new invariants of

ovoids in finite orthogonal spaces. This paper is suitable for publication in DCC provided

the following changes are made:

p3 | 49 reference to [19] should be replaced by: Charnes and Dempwolff [CD7], and this reference should be included in the bibliography. It should also be mentioned that fingerprints give an effective method of calculating the automorphism groups of ovoids.

[CD7] was added to the list of references. Included the statement "fingerprints give an effective method of calculating the automorphism groups of ovoids."

Replaced [19] by [CD7] in p3|49.

The following typos were corrected.

p13 17 and the ---> and

p13 23 Known ---> The known .. Kantor ---> the Kantor

p13 58 page [25]. ---> page [25]; Charnes and Dempwolff [CD7] calculate the automorphism groups and fingerprints of these ovoids.

[CD7]

The translation planes of order 49 and their automorphism groups

C. Charnes; U. Dempwolff.

Math. Comp. 67 (1998), 1207-1224.

Title:

On Incidence Structures of Nonsingular Points and Hyperbolic Lines of Ovoids in Finite Orthogonal Spaces

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Abstract. We study the point-line incidence structures of nonsingular points and hyperbolic secant lines associated with ovoids in finite orthogonal spaces. We show that these incidence structures frequently produce partial linear spaces and the parameters of the bipartite graphs (called *ovoidal graphs*) associated with these structures produce simple and effective isomorphism invariants to distinguish non-isomorphic ovoids. We prove explicit formulas for these isomorphism invariants for a number of infinite families of 2-transitive ovoids.

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Keywords: orthogonal space, ovoid, partial linear space, ovoidal graph, fingerprint.

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AMS Classification: 05B25, 51E20, 51E22

1. NOTATION AND DEFINITIONS

An *orthogonal space* is a pair (V, Q) such that V is a finite dimensional vector space of dimension n over $GF(q)$ and $Q : V \rightarrow GF(q)$ is a quadratic form, i.e.

$$Q(\lambda x) = \lambda^2 Q(x)$$

and

$$Q(x + y) = Q(x) + Q(y) + f(x, y),$$

for all $\lambda \in F; x, y \in V$, where $f(,)$ is a bilinear form. The quadratic form Q is *nondegenerate* if $f(x, y) = 0$ for $Q(x) = 0$ and all $y \in V$ implies $x = 0$. The *points* of an orthogonal space (V, Q) are one-dimensional subspaces and $\langle v \rangle$ is called a *singular* point if $Q(v) = 0$ and *nonsingular* otherwise. Two points $\langle v_1 \rangle$ and $\langle v_2 \rangle$ are *perpendicular* if $f(v_1, v_2) = 0$. Two sets of points X and Y are *totally nonperpendicular* if $f(x, y) \neq 0$ for any $\langle x \rangle \in X$ and for any $\langle y \rangle \in Y$. A subspace S is *totally singular* if $Q(v) = 0$ for all $v \in S$. When $n = 2m$ and Q is nondegenerate, there are two types of quadratic forms up to equivalence. One form is called *hyperbolic* which produces maximal totally singular subspaces with dimension m . The other form is called *elliptic* and it produces maximal totally singular subspaces with dimension $m - 1$. When $n = 2m - 1$ and Q is nondegenerate, there is only one type of quadratic form up to equivalence. This form is called *parabolic* which produces maximal totally singular subspaces with dimension $m - 1$. When Q is nondegenerate, (V, Q) is called an $O_{2m-1}(q)$ space if $n = 2m - 1$, or an $O_{2m}^{\pm}(q)$ space if $n = 2m$, using superscript $+$ or $-$ according as Q is hyperbolic or elliptic. Further details of these definitions can be found in Artin [1], Kantor [19] and Taylor [30].

An *r-cap* in an orthogonal space is a set of pairwise nonperpendicular singular points with cardinality r . An *ovoid* $\mathcal{O}$ in an orthogonal space of types $O_{2m}^+(q)$, $O_{2m-2}^-(q)$ or $O_{2m-1}(q)$ is a set of singular points such that every maximal totally singular subspace contains just one point in $\mathcal{O}$, or equivalently, $\mathcal{O}$ is a cap of size $q^{m-1} + 1$. Ovoids have been shown to be nonexistent in $O_{2m-2}^-(q)$ for $m \geq 4$ by Thas[33] and $O_{2m-1}(q)$ for $m > 4$ by Gunawardena and Moorhouse [18, 15]. For $m > 3$, other nonexistence results can be found in [4], [14], [25], and [29]. Ovoids in $O_6^+(q)$ are mapped via the Klein correspondence onto translation planes which have dimension 2 over its kernel $GF(q)$ [22].

The *orthogonal group* (also called *the group of isometries*) is the subgroup of $GL(V)$ which fixes the quadratic form. The *generalized orthogonal group* (also called *the group of similarities*), G , is the subgroup of $GL(V)$ which fixes the quadric (the set of singular points). The *automorphism group* of an ovoid $\mathcal{O}$, denoted $Aut(\mathcal{O})$, is defined as $Aut(\mathcal{O}) = G_{\mathcal{O}}/G_{(\mathcal{O})}$, where $G_{\mathcal{O}}$ is the stabilizer of $\mathcal{O}$ in G and $G_{(\mathcal{O})}$ is the pointwise stabilizer of $\mathcal{O}$ in G . A *primitive (2-transitive)* ovoid is an ovoid $\mathcal{O}$ whose automorphism group acts primitively (2-transitively) as a permutation group on $\mathcal{O}$. Two r -caps are

isomorphic if there is a group element in the generalized orthogonal group that takes one cap to the other.

Kleidman [20] has classified 2-transitive ovoids in polar spaces. Gunawardena [17] has extended the Kleidman's results by determining the ovoids in $O_8^+(q)$ which admits a primitive group. Bamberg and Pentilla [3] recently classified the ovoids and spreads in finite polar spaces which are stabilized by an insoluble transitive group of collineations, as a more general classification of m -systems admitting such groups. Table 1 shows the Kleidman's list of 2-transitive ovoids for the orthogonal spaces. An ovoid is called *trivial* if it contains all the singular points in a space. Note that ovoids in certain geometries automatically give rise to ovoids in larger geometries. For example, ovoids in $O_4^+(q)$ contain induced ovoids from $O_3(q)$, $O_5(q)$ contain induced ovoids from $O_4^-(q)$, and $O_6^+(q)$ contain induced ovoids from $O_4^-(q)$ and $O_5(q)$. Also $O_8^+(q)$ ovoids contain induced ovoids from $O_7(q)$.

TABLE 1. The 2-transitive ovoids in Orthogonal Spaces

Space	$Aut(\mathcal{O})$	$ \mathcal{O} $	Remarks
$O_2^+(q)$	Z_2	2	Trivial
$O_3(q)$	$Aut(PSL_2(q))$	$q + 1$	Trivial
$O_4^-(q)$	$Aut(PSL_2(q^2))$	$q^2 + 1$	Trivial
$O_4^+(q)$	$Aut(PSL_2(q))$	$q + 1$	$\lfloor \log_p(q)/2 \rfloor$ isomorphism classes [20]
$O_5(q)$	$Aut(Sz(q))$	$q^2 + 1$	$q = 2^{2h+1}, h \geq 1$, Tits [32]
$O_7(3)$	$Sp_6(2)$	28	Patterson [26, 19, 3]
$O_7(q)$	$Aut(^2G_2(q))$	$q^3 + 1$	$q = 3^{2h+1}, h \geq 1$, Ree-Tits [31]
$O_7(q)$	$Aut(PSU_3(q))$	$q^3 + 1$	$q = 3^h, h \geq 2$, Thas-Kantor [5, 19]
$O_8^+(2)$	S_9	9	Dye [12, 19]
$O_8^+(q)$	$Aut(PSU_3(q))$	$q^3 + 1$	$q \geq 5$ and $q \equiv 2 \pmod{3}$, Kantor [19]
$O_8^+(q)$	$Aut(PSL_2(q^3))$	$q^3 + 1$	q is even and $q \geq 4$, Kantor [19]

Let $\mathcal{P}$ be a finite non-empty set of elements called points, $\mathcal{L}$ be a family of subsets of $\mathcal{P}$ called lines, and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$ be an incidence relation. Let s, t be two nonnegative integers. An incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ has order (s, t) if $\mathcal{I}$ satisfies the following axioms: (a) Any two distinct points are incident with at most one line; (b) Each line is incident with exactly $s + 1$ points; (c) Each point is incident with exactly $t + 1$ lines. If both $s, t \geq 1$ then $\mathcal{S}$ is called a *partial linear space of order (s, t)* [12]. Otherwise $\mathcal{S}$ is called a *trivial structure*.

2. A SURVEY OF INVARIANTS

Our discussion is limited to the invariants which can be computed for orthogonal ovoids. An invariant is *complete* for a given orthogonal space if it distinguishes any two non-isomorphic ovoids in the space. An effective isomorphism invariant must be complete and it should be fast to compute. Here we survey various isomorphic invariants used in recent classifications of ovoids and translation planes.

The invariant known as *fingerprint*, introduced by J. H. Conway, is used by Charnes [6] to identify translation planes of order 25 (i.e., corresponding to $O_6^+(5)$ ovoids) which have been previously classified by Czerwinski and Oakden [11]. Fingerprints are complete for ovoids in $O_6^+(q)$, $q = 3, 5, 7$, and give an effective method of calculating the automorphism groups of ovoids, see Charnes and Dempwolff [7]. In another paper, Charnes and Dempwolff [8] use fingerprints as an aid to classify $O_8^+(5)$ ovoids. Moorhouse in [24] describes a simple way of computing fingerprints which has $O(r^3)$ time complexity for an r -cap and shows its connection to two-graphs of ovoids. The ovoids with regular two-graphs have the same fingerprint. Since ovoids in $O_{2m-1}(q)$ spaces have regular two-graphs, see Gunawardena and Moorhouse [18], fingerprints do not separate non-isomorphic ovoids in $O_{2m-1}(q)$. Also, fingerprints do not separate ovoids when the field $GF(q)$ has even characteristic.

Mathon and Royle [21] use another invariant called *regulus profile* in their classification of translation planes of order 49. They create short and long profiles using the intersections of reguli with a spread. For ovoids, we can create these profiles using conics which are $O_3(q)$ subspaces. Their long

profiles are complete for $O_6^+(7)$ ovoids but they are more expensive to compute compared to fingerprints. These profiles have advantages over fingerprints since they can be used in even characteristic cases and in any dimension.

3. THE POINT-LINE INCIDENCE STRUCTURE OF NONSINGULAR POINTS AND HYPERBOLIC LINES OF AN OVOID AND A NEW INVARIANT

In this section, we introduce a new invariant that can be used in any characteristic field or in any dimension. Our invariant helps visualize the nonsingular point hyperbolic line incidence structure of an ovoid and has a faster computing time than that of fingerprints. It is complete for the classifications mentioned in Section 2.

Let $\mathcal{O}$ be an ovoid in $O_{2m}^+(q)$, $O_{2m-1}(q)$ or $O_{2m-2}^-(q)$. We can write $\mathcal{O}$ as a set of singular points $\{\langle v_1 \rangle, \langle v_2 \rangle, \dots, \langle v_r \rangle\}$, where $r = q^{m-1} + 1$. Let $\mathcal{HL}$ denote the set of hyperbolic lines defined by $\{\langle v_i, v_j \rangle \mid 1 \leq i < j \leq r\}$. Thus $|\mathcal{HL}| = r(r-1)/2$. Now we define incidence structures $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, $\mathcal{S}_{sq} = (\mathcal{SQ}, \mathcal{L}_{sq}, \mathcal{I}_{sq})$, and $\mathcal{S}_{nsq} = (\mathcal{NSQ}, \mathcal{L}_{nsq}, \mathcal{I}_{nsq})$ as follows. For $1 \leq i < j \leq r$, let the line ℓ_{ij} be the set containing $q-1$ nonsingular points incident with the hyperbolic line $\langle v_i, v_j \rangle$ given by $\{\langle v_i + \alpha v_j \rangle \mid \alpha \neq 0\}$. Let $\mathcal{P}$ be the set of nonsingular points given by $\cup_{1 \leq i < j \leq r} \ell_{ij}$ and $\mathcal{L}$ be the set of lines given by $\{\ell_{ij} \mid 1 \leq i < j \leq r\}$. Thus $|\mathcal{L}| \leq r(r-1)/2$. The incidence relation $\mathcal{I}$ is the subset of $\mathcal{P} \times \mathcal{L}$ such that $(p, \ell) \in \mathcal{I}$ if and only if $p \in \ell$. Let $\mathcal{SQ}$ ($\mathcal{NSQ}$) be the set of all the points $\langle x \rangle \in \mathcal{P}$ such that $Q(x)$ is a square (nonsquare) in the field $GF(q)$ (Note that square(nonsquare) is well defined since $Q(\lambda x) = \lambda^2 Q(x)$ for any $\lambda \in GF(q)$). Clearly, $\mathcal{SQ}$ and $\mathcal{NSQ}$ partition $\mathcal{P}$. Let $\mathcal{L}_{sq}$ be $\{\ell \cap \mathcal{SQ} \mid \ell \in \mathcal{L}\}$ and $\mathcal{I}_{sq}$ be $\{(p, \ell) \in (\mathcal{SQ} \times \mathcal{L}_{sq}) \mid p \in \ell\}$. Let $\mathcal{L}_{nsq}$ be $\{\ell \cap \mathcal{NSQ} \mid \ell \in \mathcal{L}\}$ and $\mathcal{I}_{nsq}$ be $\{(p, \ell) \in (\mathcal{NSQ} \times \mathcal{L}_{nsq}) \mid p \in \ell\}$.

Now we construct a bipartite graph, called the *ovoidal graph* $\mathcal{OG}$ between the two sets $\mathcal{HL}$ and $\mathcal{P}$. The set of vertices of $\mathcal{OG}$ is $\mathcal{HL} \cup \mathcal{P}$ and there is an edge between the vertices $h \in \mathcal{HL}$ and $p \in \mathcal{P}$ if and only if the nonsingular point p is incident with the hyperbolic line h . The ovoidal graph $\mathcal{OG}$ is a (δ, γ) -regular bipartite graph if every $h \in \mathcal{HL}$ has degree δ and every $p \in \mathcal{P}$ has degree γ . Clearly the degree of any vertex in $\mathcal{HL}$ is $q-1$. Let D_{sq} and D_{nsq} are the multisets of the degrees of the vertices of $\mathcal{SQ}$ and $\mathcal{NSQ}$ constructed as follows. Let $T = \{d_1, d_2, \dots, d_k\}$, where d_i are positive integers, be the set of all possible degrees of vertices in $\mathcal{P}$. Let $d \in T$ and $C_{sq}(d) = |\{x \in \mathcal{SQ} \mid \deg(x) = d\}|$ and $C_{nsq}(d) = |\{x \in \mathcal{NSQ} \mid \deg(x) = d\}|$. Let $D_{sq} = \{(d, C(d)) \mid d \in T, C(d) = C_{sq}(d) \geq 1\}$ and $D_{nsq} = \{(d, C(d)) \mid d \in T, C(d) = C_{nsq}(d) \geq 1\}$. Since the incidence structure $\mathcal{S}$, the set $\{\mathcal{S}_{sq}, \mathcal{S}_{nsq}\}$, and the graph $\mathcal{OG}$ are isomorphism invariants of $\mathcal{O}$, we select the list $Inv(\mathcal{O}) = (|\mathcal{P}|, \{(|\mathcal{SQ}|, D_{sq}), (|\mathcal{NSQ}|, D_{nsq})\})$ as our isomorphism invariant for the ovoid. Of course if $GF(q)$ has even characteristic, $\mathcal{P} = \mathcal{SQ}$, $D = D_{sq}$ and $\mathcal{NSQ} = \emptyset$. The reader should note that if $Q(x)$ is replaced by $\epsilon Q(x)$ where ϵ is a nonsquare element in $GF(q)$, then nonsquare points will become square points and, vice versa. Hence the incidence structures $\mathcal{S}_{sq}$ and $\mathcal{S}_{nsq}$ are not invariant under the group of similarities but the set $\{\mathcal{S}_{sq}, \mathcal{S}_{nsq}\}$ is.

The invariants of the ovoidal graph can be computed in $O(r^2 q \log(r^2 q))$ time complexity. Let $\mathcal{P} = \{\langle x_1 \rangle, \langle x_2 \rangle, \dots, \langle x_M \rangle\}$ where each $\langle x_i \rangle$ is a nonsingular point. We normalize each representative vector x_i by making its first nonzero entry to be 1. It is also clear that $M \leq (q-1)r(r-1)/2$. The authors utilized a heap sort with lexicographical ordering to calculate the invariants of the ovoidal graph. Each heap element contains a nonsingular point x_i , a flag to show square or nonsquare, and a counter to calculate its degree. We insert the $q-1$ nonsingular points in $\mathcal{P}$ incident with each $h_{ij} \in \mathcal{HL}$ for $1 \leq i < j \leq r$ into the heap as follows. When a nonsingular point x_i is not in the heap it will be inserted as a new element in the heap with the counter value 1 and set the flag to square (or nonsquare) depending on x_i is square (or nonsquare). Otherwise we just increment the counter. We can insert all the generated $(q-1)r(r-1)/2$ nonsingular points in $O(r^2 q \log(r^2 q))$ time resulting in a heap of size M . When we remove elements from the heap, the number of distinct elements of $\mathcal{P}$ ($= M$), $\mathcal{SQ}$ and $\mathcal{NSQ}$ as well as their respective degrees can be counted. Removing all the elements from the heap takes $O(M \log M)$ time.

4. THE POINT-LINE INCIDENCE STRUCTURES OF 2-TRANSITIVE OVOIDS

The following theorem presents the point-line incidence structures $\mathcal{S}, \mathcal{S}_{sq}, \mathcal{S}_{nsq}$, and the ovoidal graph $\mathcal{OG}$ for some of the 2-transitive ovoids in Table 1.

Theorem 4.1. *The following statements hold for the 2-transitive ovoids mentioned below.*

1. Let $\mathcal{O}$ be an $O_2^+(q)$ ovoid. Then $\mathcal{S}$ is an incidence structure with parameters $s = q - 2$ and $t = 0$. The ovoidal graph $\mathcal{OG}$ is (δ, γ) -regular with $\delta = q - 1, \gamma = 1$. When q is odd, $\mathcal{S}_{sq}$ and $\mathcal{S}_{nsq}$ are incidence structures with parameters $s_{sq} = s_{nsq} = (q - 3)/2$ and $t_{sq} = t_{nsq} = 0$. The subgraphs of $\mathcal{OG}$ induced by $\mathcal{SQ}$ and $\mathcal{NSQ}$ are (δ, γ) -regular with $\delta = (q - 1)/2, \gamma = 1$.
2. Let $\mathcal{O}$ be an $O_3(q)$ ovoid. If $q(> 2)$ is even then $\mathcal{S}$ is a partial linear space with parameters $s = q - 2$ and $t = (q - 2)/2$. If $q = 2 (s = 0)$, then the ovoidal graph $\mathcal{OG}$ is (δ, γ) -regular with $\delta = 1, \gamma = 1$. If $q(> 3)$ is odd, then $\mathcal{S}_{sq}$ and $\mathcal{S}_{nsq}$ are partial linear spaces with parameters $s_{sq} = t_{sq} = (q - 3)/2$ and $s_{nsq} = (q - 3)/2, t_{nsq} = (q - 1)/2$. If $q = 3$, then the subgraph of $\mathcal{OG}$ induced by $\mathcal{SQ}$ is (δ, γ) -regular with $\delta = 1, \gamma = 1$, and the subgraph of $\mathcal{OG}$ induced by $\mathcal{NSQ}$ is (δ, γ) -regular with $\delta = 1, \gamma = 2$.
3. Let $\mathcal{O}$ be an $O_4^-(q)$ elliptic ovoid. If $q > 2$ then $\mathcal{S}$ is a partial linear space with $s = q - 2$ and $t = (q^2 - q - 2)/2$. If $q = 2$ then the ovoidal graph $\mathcal{OG}$ is (δ, γ) -regular with $\delta = 1, \gamma = 1$. If $q(> 3)$ is odd, then $\mathcal{S}_{sq}$ and $\mathcal{S}_{nsq}$ are partial linear spaces with parameters $s_{sq} = s_{nsq} = (q - 3)/2$ and $t_{sq} = t_{nsq} = (q^2 - q - 2)/2$. If $q = 3$, then the subgraphs of $\mathcal{OG}$ induced by $\mathcal{SQ}$ and $\mathcal{NSQ}$ are (δ, γ) -regular with $\delta = 1, \gamma = 1$.
4. Let $\mathcal{O}$ be the Tits ovoid in the space $O_5(q)$ where $q = 2^{2h+1}, h > 0$. Then $\mathcal{S}$ is a partial linear space with $s = q - 2$, and $t = (q - 2)/2$.
5. Let $\mathcal{O}$ be an ovoid in $O_7(3)$. Then the subgraph of $\mathcal{OG}$ induced by $\mathcal{SQ}$ is (δ, γ) -regular with $\delta = 1, \gamma = 1$, and the subgraph of $\mathcal{OG}$ induced by $\mathcal{NSQ}$ is (δ, γ) -regular with $\delta = 1, \gamma = 6$.
6. Let $q = 3^h$ where $h \geq 2$ and $\mathcal{O}$ be the Thas-Kantor ovoid in the parabolic space $O_7(q)$. Then $\mathcal{S}_{sq}$ is a partial linear space with $s_{sq} = (q - 3)/2, t_{sq} = (q - 3)/2$, and $\mathcal{S}_{nsq}$ is a partial linear space with $s_{nsq} = (q - 3)/2$, and $t_{nsq} = (3q + 1)/2$.
7. Let $\mathcal{O}$ be an ovoid in $O_8^+(2)$. Then the ovoidal graph $\mathcal{OG}$ is (δ, γ) -regular with $\delta = 1, \gamma = 1$.
8. Let $\mathcal{O}$ be a Kantor $PSL(2, q^3)$ ovoid where $q = 2^h, h \geq 2$. Then $\mathcal{S}$ is an incidence structure with parameters $s = (q - 2)$ and $t = 0$.

Furthermore, the ovoidal graph invariants of the above ovoids are as shown in Table 2.

We use the following lemmas to prove Theorem 4.1.

Lemma 4.2. *Let $\mathcal{O}$ be 2-transitive in each of the following cases and $|\mathcal{O}| = r$. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, $\mathcal{S}_{sq} = (\mathcal{SQ}, \mathcal{L}_{sq}, \mathcal{I}_{sq})$, and $(\mathcal{NSQ}, \mathcal{L}_{nsq}, \mathcal{I}_{nsq})$ be incidence structures associated with $\mathcal{O}$ as defined above. Then $\text{Inv}(\mathcal{O}) = (|\mathcal{P}|, \{(|\mathcal{SQ}|, D_{sq}), (|\mathcal{NSQ}|, D_{nsq})\})$ satisfies the following.*

1. If $\mathcal{S}$ is of order (s, t) with $s \geq 1$ then $q > 2$ and $\mathcal{OG}$ is (δ, γ) -regular with $\delta = q - 1, \gamma = t + 1$ and $|\mathcal{P}| = (q - 1)r(r - 1)/(2(t + 1))$. If q is odd then $|\mathcal{SQ}| = |\mathcal{NSQ}|$ and $D_{sq} = D_{nsq} = \{(t + 1, |\mathcal{P}|/2)\}$. If $q(> 2)$ is even $D_{sq} = \{(t + 1, |\mathcal{P}|)\}$. If $q = 2$ (i.e., $s = 0$) and $\mathcal{OG}$ is (δ, γ) -regular then $\delta = 1, \gamma = r(r - 1)/(2|\mathcal{P}|)$ and $D_{sq} = \{(\gamma, |\mathcal{P}|)\}$.
2. If q is odd and $\mathcal{S}_{sq}$ is of order (s_{sq}, t_{sq}) with $s_{sq} \geq 1$ then $q > 3, s_{sq} = (q - 3)/2, |\mathcal{SQ}| = (q - 1)r(r - 1)/(4(t_{sq} + 1))$ and $D_{sq} = \{(t_{sq} + 1, |\mathcal{SQ}|)\}$. If $q = 3$ (i.e., $s_{sq} = 0$) and every square point in $\mathcal{OG}$ has a degree γ then $\gamma = r(r - 1)/(2|\mathcal{SQ}|)$ and $D_{sq} = \{(\gamma, |\mathcal{SQ}|)\}$.
3. If $\mathcal{S}_{nsq}$ is of order (s_{nsq}, t_{nsq}) with $s_{nsq} \geq 1$ then $q > 3, s_{nsq} = (q - 3)/2, |\mathcal{NSQ}| = (q - 1)r(r - 1)/(4(s_{nsq} + 1))$ and $D_{nsq} = \{(t_{nsq} + 1, |\mathcal{NSQ}|)\}$. If $q = 3$ (i.e., $s_{nsq} = 0$) and every non-square point in $\mathcal{OG}$ has a degree γ then $\gamma = r(r - 1)/(2|\mathcal{NSQ}|)$ and $D_{nsq} = \{(\gamma, |\mathcal{NSQ}|)\}$.

Proof. Assume $\mathcal{S}$ is of order (s, t) with $s \geq 1$. Since $s = q - 2$, we get $q > 2$. Since $\mathcal{O}$ is 2-transitive and $s \geq 1$, $|\mathcal{L}| = |\mathcal{HL}| = r(r - 1)/2$ and any $h \in \mathcal{HL}$ can be uniquely identified by an $\ell \in \mathcal{L}$. Thus $\mathcal{OG}$ is a regular bipartite graph with $\delta = q - 1, \gamma = t + 1$ and counting $(p, \ell) \in \mathcal{I}$ in two ways we get $|\mathcal{P}| = (q - 1)r(r - 1)/(2(t + 1))$. Assume q is odd. Since $t_{sq} = t_{nsq} = t$ and $s_{sq} = s_{nsq} = (q - 3)/2$,

TABLE 2. Ovoidal Graph Invariants for some of the 2-transitive ovoids

Space	$Aut(\mathcal{O})$	q	$ \mathcal{O} $	$ \mathcal{P} $	$ \mathcal{SQ} $	$ \mathcal{NSQ} $	D_{sq}	D_{nsq}
$O_2^+(q)$	Z_2	even	2	$q-1$	$ \mathcal{P} $		$\{(1, \mathcal{P})\}$	
$O_2^+(q)$	Z_2	odd	2	$q-1$	$\frac{q-1}{2}$	$\frac{q-1}{2}$	$\{(1, \mathcal{SQ})\}$	$\{(1, \mathcal{NSQ})\}$
$O_3(q)$	$Aut(PSL_2(q))$	even	$q+1$	q^2-1	$ \mathcal{P} $		$\{(\frac{q}{2}, \mathcal{P})\}$	
$O_3(q)$	$Aut(PSL_2(q))$	odd	$q+1$	q^2	$\frac{q^2+q}{2}$	$\frac{q^2-q}{2}$	$\{(\frac{q-1}{2}, \mathcal{SQ})\}$	$\{(\frac{q+1}{2}, \mathcal{NSQ})\}$
$O_4^-(q)$	$Aut(PSL_2(q^2))$	even	q^2+1	q^3+q	$ \mathcal{P} $		$\{(q^2-q, \mathcal{P})\}$	
$O_4^-(q)$	$Aut(PSL_2(q^2))$	odd	q^2+1	q^3+q	$\frac{q^3+q}{2}$	$\frac{q^3-q}{2}$	$\{(\frac{q^2-q}{2}, \mathcal{SQ})\}$	$\{(\frac{q^2+q}{2}, \mathcal{NSQ})\}$
$O_5(q)$	$Aut(Sz(q))$	$2^{2h+1} \geq 8$	q^2+1	$\frac{(q^2-q)(q^2+1)}{2}$	$ \mathcal{P} $		$\{\frac{q}{2}, \mathcal{P} \}$	
$O_7(3)$	$Sp_6(2)$	3	28	441	378	63	$\{(1, 378)\}$	$\{(6, 63)\}$
$O_7(q)$	$Aut(PSU_3(q))$	$3^h \geq 9$	q^3+1	$\frac{(q^6+q^3)(2q+1)}{3(q+1)}$	$\frac{q^6+q^3}{2}$	$\frac{(q^6+q^3)(q-1)}{6(q+1)}$	$\{(\frac{q-1}{2}, \mathcal{SQ})\}$	$\{(\frac{3(q+1)}{2}, \mathcal{NSQ})\}$
$O_8^+(2)$	S_9	2	9	36	36		$\{(1, 36)\}$	
$O_8^+(q)$	$Aut(PSL_2(q^3))$	$2^h \geq 4$	q^3+1	$\frac{q^3(q^3+1)(q-1)}{2}$	$ \mathcal{P} $		$\{(1, \mathcal{P})\}$	

we get $|\mathcal{SQ}| = |\mathcal{NSQ}|$ and $D_{sq} = D_{nsq} = \{(t+1, |\mathcal{P}|/2)\}$. Assume $q(> 2)$ is even. Since $|\mathcal{SQ}| = |\mathcal{P}|$, we get $D_{sq} = \{(t+1, |\mathcal{P}|)\}$. Assume $q = 2$ (i.e., $s = 0$) and $\mathcal{OG}$ is (δ, γ) -regular. Since $\delta = s+1$ and $|\mathcal{HL}|\delta = \gamma|\mathcal{P}|$, we get $\delta = 1, \gamma = r(r-1)/(2|\mathcal{P}|)$ and $D_{sq} = \{(\gamma, |\mathcal{P}|)\}$. Thus (1) holds.

To prove (2), assume the case where q is odd and $\mathcal{S}_{sq}$ is of order (s_{sq}, t_{sq}) with $s_{sq} \geq 1$. Since $s_{sq} = (q-3)/2$, we get $q > 3$. Assume $q(> 3)$ is odd. By counting $(p, \ell) \in \mathcal{I}_{sq}$ in two ways we get $|\mathcal{SQ}| = (q-1)r(r-1)/(4(t_{sq}+1))$ and $D_{sq} = \{(t_{sq}+1, |\mathcal{SQ}|)\}$. Assume $q = 3$ (i.e., $s_{sq} = 0$) and every square point in $\mathcal{OG}$ has a degree γ . Since $|\mathcal{HL}|(s_{sq}+1) = \gamma|\mathcal{SQ}|$, we get $\gamma = r(r-1)/(2|\mathcal{SQ}|)$ and $D_{sq} = \{(\gamma, |\mathcal{SQ}|)\}$.

The proof of (3) is similar to the proof of (2). ■

4.1. Tits's Ovoid in $O_5(q)$, $q(\geq 8)$ is even, stabilized by $Aut(Sz(q))$. We now describe the point-line incidence structure of the hyperbolic lines of the Tits ovoids [32] in $O(5, q)$ when $q = 2^{2h+1}$, $(h > 0)$. Let the quadratic form Q be defined by

$$Q(x_1, x_2, x_3, x_4, x_5) = x_1x_5 + x_2x_4 + x_3^2.$$

Tits [32] has shown that the automorphism group of the Tits ovoid is 2-transitive. It is known, see [35], that the Tits ovoid can be described as the set of points of the form:

$$T = \{\langle(0, 0, 0, 0, 1)\rangle\} \cup \{\langle(1, x, y, x^{\alpha+1} + y^{\alpha}, y^2 + x^{\alpha+2} + xy^{\alpha+1})\rangle\}$$

where $\alpha \in Aut(GF(q))$ satisfies $a^{\alpha^2} = a^2$ for all $a \in GF(q)$.

Let $GF(q)$ and α be as in the previous paragraph and let $\bar{F} = GF(q) \setminus \{0, 1\}$. We need the following technical lemma and corollary.

Lemma 4.3. *The function*

$$\theta : \bar{F} \rightarrow \bar{F} : b \rightarrow \frac{b}{b^{\alpha} + b}$$

is a bijection.

Proof. First note that if $b^{\alpha} + b = 0$, then $b^{\alpha-1} = 1$ since $b \neq 0$. Therefore, $1 = b^{(\alpha-1)(\alpha+1)} = b$. Also if $\frac{b}{b^{\alpha} + b} = 1$, then $b = b^{\alpha} + b$. Hence $b^{\alpha} = 0$ and $b = 0$. Therefore θ is a well-defined function.

We will show that θ is a bijection by showing $\frac{b}{b^{\alpha} + b} = \frac{c}{c^{\alpha} + c}$ if and only if $b = c$. One direction is clear. Assume that $\frac{b}{b^{\alpha} + b} = \frac{c}{c^{\alpha} + c}$. Then $bc^{\alpha} + bc = b^{\alpha}c + bc$ and $b^{\alpha-1} = c^{\alpha-1}$. Raising both sides to the $\alpha+1$ power yields $b = c$. Thus θ is a bijection. ■

Corollary 4.4. *The function*

$$\phi : \bar{F} \rightarrow \bar{F} : a \rightarrow \frac{a+1}{a^\alpha + a}$$

is a bijection.

Proof. Note that $\frac{a+1}{a^\alpha + a} = \frac{a+1}{(a^\alpha+1)+(a+1)}$. Hence the result follows from Lemma 4.3 where $b = a + 1$. ■

The following lemma determines the point-line incidence structure for a fixed hyperbolic line of the ovoid.

Lemma 4.5. *Let*

$$\ell = \langle \langle (1, 0, 0, 0, 0) \rangle, \langle (0, 0, 0, 0, 1) \rangle \rangle.$$

Then every nonsingular point P on the line ℓ is incident with $q/2$ hyperbolic lines of the ovoid $\mathcal{O}$.

Proof. Let P be a nonsingular point on the line ℓ . Then $P = \langle (1, 0, 0, 0, t) \rangle$ where $t \neq 0$. Any hyperbolic line of the ovoid (other than ℓ) that is incident with P must be of the form

$$(1) \quad \langle \langle (1, x, y, x^{\alpha+1} + y^\alpha, y^2 + x^{\alpha+2} + xy^\alpha) \rangle, \langle (1, u, v, u^{\alpha+1} + v^\alpha, v^2 + u^{\alpha+2} + uv^\alpha) \rangle \rangle$$

where x and y are both not zero. This will occur if and only if there is a nonzero value $a \in GF(q)$ such that

$$\langle (1, 0, 0, 0, t) \rangle = \langle (1, x, y, x^{\alpha+1} + y^\alpha, y^2 + x^{\alpha+2} + xy^\alpha) \rangle + a \langle (1, u, v, u^{\alpha+1} + v^\alpha, v^2 + u^{\alpha+2} + uv^\alpha) \rangle.$$

From this equation we have the following equations:

$$(2) \quad x = au$$

$$(3) \quad y = av$$

$$(4) \quad x^{\alpha+1} + y^\alpha = au^{\alpha+1} + av^\alpha$$

$$(5) \quad (1+a)t = y^2 + x^{\alpha+2} + xy^\alpha + av^2 + au^{\alpha+2} + auv^\alpha.$$

Note that $a \neq 1$. Using equations (2) and (3) in equation (4), we have

$$\begin{aligned} x^{\alpha+1} + y^\alpha &= (au)^{\alpha+1} + (av)^\alpha \\ &= a^{\alpha+1}u^{\alpha+1} + a^\alpha v^\alpha. \end{aligned}$$

Consequently from equation (4) we have

$$a^{\alpha+1}u^{\alpha+1} + a^\alpha v^\alpha = au^{\alpha+1} + av^\alpha$$

or

$$(6) \quad v^\alpha = \frac{a(a+1)^\alpha u^{\alpha+1}}{a^\alpha + a}.$$

Since $b^{\alpha^2} = b^2$ for all $b \in GF(q)$, it also follows that

$$(7) \quad v^2 = \frac{a^\alpha(a+1)^2 u^{\alpha+2}}{a^\alpha + a^2}.$$

Using equations (6) and (7) as well as equations (2) and (3) in equation (5), we have

$$\begin{aligned}
 y^2 + x^{\alpha+2} + xy^\alpha + av^2 + au^{\alpha+2} + auv^\alpha &= a^2v^2 + a^{\alpha+2}u^{\alpha+2} + (au)a^\alpha v^\alpha + av^2 + au^{\alpha+2} + auv^\alpha \\
 &= (a^2 + a)v^2 + (a^{\alpha+2} + a)u^{\alpha+2} + a(a^\alpha + 1)uv^\alpha \\
 &= \frac{a^\alpha(a^2 + a)(a^2 + 1)}{a^\alpha + a^2}u^{\alpha+2} \\
 &\quad + (a^{\alpha+2} + a)u^{\alpha+2} + \frac{a^2(a^\alpha + 1)^2}{a^\alpha + a}u^{\alpha+2} \\
 &= \frac{(a + 1)^{\alpha+2}}{(a^\alpha + a^2)(a^\alpha + a)}(au)^{\alpha+2} \\
 &= \frac{(a + 1)^{\alpha+2}}{(a^\alpha + a)^\alpha(a^\alpha + a)}a^{\alpha+2}u^{\alpha+2}.
 \end{aligned}$$

Hence $t(a + 1) = \frac{(a+1)^{\alpha+2}}{(a^\alpha + a)^{\alpha+1}}(au)^{\alpha+2}$. Consequently

$$(8) \quad t = \frac{(a + 1)^{\alpha+1}}{(a^\alpha + a)^{\alpha+1}}(au)^{\alpha+2} = \frac{(a + 1)^{\alpha+1}}{(a^\alpha + a)^{\alpha+1}}x^{\alpha+2}.$$

Note that $x^{\alpha+2} \neq t$ and $u^{\alpha+2} \neq t$. If $x^{\alpha+2} = t$, we would have $\frac{(a+1)^{\alpha+1}}{(a^\alpha + a)^{\alpha+1}} = 1$. Raising both sides to the $\alpha - 1$ power gives us $\frac{a+1}{a^\alpha + a} = 1$. Hence $a + 1 = a^\alpha + a$ or $a^\alpha = 1$. Thus $a = 1$ which is impossible. A similar argument shows that $u^{\alpha+2} \neq t$.

We can now count the number of hyperbolic lines in the ovoidal graph that are incident with the nonsingular point $P = \langle(1, 0, 0, 0, t)\rangle$. For each x such that $x^{\alpha+2} \neq t$, $\frac{t}{x^{\alpha+2}} \notin \{0, 1\}$. By Corollary 4.7, there is a unique $a \in GF(q) \setminus \{0, 1\} = \bar{F}$ such that $\frac{a+1}{a^\alpha + a} = (\frac{t}{x^{\alpha+2}})^{\alpha-1}$. Let u satisfy the equation $x = au$. From the previous paragraph $u^{\alpha+2} \neq t$. Thus x uniquely determines a and u . Since there are $q - 2$ choices for x , there are $q - 2$ choices for the ordered pairs (x, u) that satisfy equation (8). By equation (6), v is determined by a, u and it follows from equation (3) that y is also determined. Since each ordered pair (x, u) and (u, x) determine the same hyperbolic line, there are $(q - 2)/2$ lines of the form (1) that are incident with the nonsingular point P . Since ℓ is also a hyperbolic line in the ovoidal graph incident with P , every nonsingular point in the ovoidal graph is incident with $q/2$ hyperbolic lines of the ovoid. $\blacksquare$

4.2. Thas-Kantor Ovoid in $O_7(q)$, $q = 3^h, h \geq 2$, stabilized by $Aut(PSU_3(q))$. The Thas-Kantor ovoids of $O_7(q)$ where $q = 3^h, h > 1$ is another of family 2-transitive ovoids[20]. Let the quadratic form Q be defined by

$$Q(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_1x_7 + x_2x_6 + x_3x_5 + x_4^2.$$

By [35], the Thas-Kantor ovoid can be described as

$$\mathcal{O} = \{\langle(0, 0, 0, 0, 0, 0, 1)\rangle\} \cup \{\langle(1, x, y, z, f_1(x, y, z), f_2(x, y, z), -y^2 - xf_2(x, y, z) - yf_1(x, y, z))\rangle\}.$$

where $x, y, z \in GF(q)$, n is a fixed nonsquare in $GF(q)$, $f_1(x, y, z) = x^2y - ny^3 + 2xz$, $f_2(x, y, z) = -\frac{1}{n}x^3 + xy^2 - 2yz$. We will determine the number of hyperbolic lines of the ovoid that are incident with the line

$$\ell = \langle(1, 0, 0, 0, 0, 0, 0)\rangle, \langle(0, 0, 0, 0, 0, 0, 1)\rangle.$$

Each of the nonsingular points incident with ℓ are of the form $\langle(1, 0, 0, 0, 0, 0, t)\rangle$ for $t \neq 0$.

The point $P = \langle(1, 0, 0, 0, 0, 0, t)\rangle$ for $t \neq 0$ is incident with the hyperbolic line ℓ and is also incident with hyperbolic lines in the ovoid provided that

$$(9) \quad \langle(1, 0, 0, 0, 0, 0, t)\rangle = S + aT$$

where $a \neq 0$ and

$$\begin{aligned}
 S &= \langle(1, x, y, z, f_1(x, y, z), f_2(x, y, z), -y^2 - xf_2(x, y, z) - yf_1(x, y, z))\rangle \\
 T &= \langle(1, u, v, w, f_1(u, v, w), f_2(u, v, w), -v^2 - uf_2(u, v, w) - vf_1(u, v, w))\rangle
 \end{aligned}$$

From this equation, we have the following equations:

$$(10) \quad x = -au$$

$$(11) \quad y = -av$$

$$(12) \quad z = -aw$$

$$(13) \quad f_1(x, y, z) = -af_1(u, v, w)$$

$$(14) \quad f_2(x, y, z) = -af_2(u, v, w)$$

$$(15) \quad (1+a)t = -z^2 - yf_1(x, y, z) - xf_2(x, y, z) + a(-w^2 - vf_1(u, v, w) - uf_2(u, v, w)).$$

Note that $a \neq -1$ for in that case $x = u, y = v, z = w$ and the hyperbolic line degenerates into a point. Using equations (10), (11) and (12) in equation (13), we have

$$\begin{aligned} 0 &= f_1(-au, -av, -aw) + af_1(u, v, w) \\ &= -au^2v + na^3v^3 + 2a^2uw + au^2v - anv^3 + 2a^2uw \\ &= (-a^3 + a)u^2v + n(a^3 - a)v^3 + 2(a^2 + a)uw. \end{aligned}$$

Hence it follows that

$$(-a^2 + 1)u^2v + n(a^2 - 1)v^3 + 2(a + 1)uw = 0.$$

and

$$(16) \quad (1 - a)u^2v + n(1 - a)v^3 - uw = 0$$

since $a \neq -1$. Similarly, using (10), (11) and (12) in equation (14), we have

$$\begin{aligned} 0 &= f_2(-au, -av, -aw) + af_2(u, v, w) \\ &= \frac{1}{n}a^3u^3 - a^3uv^2 - 2a^2vw - \frac{1}{n}au^3 + auv^2 - 2avw \\ &= \frac{1}{n}(a^3 - a)u^3 + (-a^3 + a)uv^2 - 2(a^2 + a)vw \end{aligned}$$

It follows that

$$\frac{1}{n}(a^2 - 1)u^3 + (-a^2 + 1)uv^2 - 2(a + 1)vw = 0$$

and

$$(17) \quad \frac{1}{n}(a - 1)u^3 + (1 - a)uv^2 + vw = 0$$

since $a \neq -1$.

Lemma 4.6. *If both u and v are not zero in formula (9), then $a = 1$ and $w = 0$.*

Proof. Consider the two equations (16) and (17) from above. Add equation (16) multiplied by v to equation (17) multiplied by u to eliminate w and obtain

$$\begin{aligned} 2(1 - a)u^2v^2 + n(1 - a)v^4 + \frac{1}{n}(a - 1)u^4 &= 0 \\ (a - 1)[nv^4 + \frac{1}{n}u^4 - 2u^2v^2] &= 0 \\ \frac{a - 1}{n}[n^2v^4 - 2nu^2v^2 + u^4] &= 0 \\ \frac{a - 1}{n}[nv^2 - u^2]^2 &= 0. \end{aligned}$$

Since both u and v are not zero, $nv^2 - u^2 \neq 0$. Hence it follows that $a - 1 = 0$ and $a = 1$.

To show that $w = 0$ consider the following cases.

Case 1: Let $u \neq 0$. It follows that

$$\begin{aligned}
 0 &= f_1(x, y, z) + f_1(u, v, w) \\
 &= f_1(-u, -v, -w) + f_1(u, v, w) \\
 &= -u^2v + nv^3 + 2uw + u^2v - nv^3 + 2uw \\
 &= 4uw.
 \end{aligned}$$

Hence it follows that $w = 0$ since $u \neq 0$.

Case 2: Let $v \neq 0$. Then using f_2 we have:

$$\begin{aligned}
 0 &= f_2(x, y, z) + f_2(u, v, w) \\
 &= f_2(-u, -v, -z) + f_2(u, v, w) \\
 &= \frac{1}{n}u^3 - uv^2 + 2vw - \frac{1}{n}u^3 + uv^2 + 2vw \\
 &= 4vw.
 \end{aligned}$$

Therefore, it follows that $w = 0$. ■

We will count the number of hyperbolic lines in the ovoidal graph that are incident with the point $\langle(1, 0, 0, 0, 0, 0, t)\rangle$ for $t \neq 0$ when both u and v are both non-zero. By Lemma 4.6, we have

$$\begin{aligned}
 \langle(1, 0, 0, 0, 0, 0, t)\rangle &= \langle(1, x, y, 0, f_1(x, y, 0), f_2(x, y, 0), -yf_1(x, y, 0) - xf_2(x, y, 0))\rangle \\
 &\quad \langle(1, u, v, 0, f_1(u, v, 0), f_2(u, v, 0), -vf_1(u, v, 0) - uf_2(u, v, 0))\rangle \\
 &= \langle(1, -u, -v, 0, -u^2v + nv^3, \frac{1}{n}u^3 - uv^2, -u^2v^2 + nv^4 + \frac{1}{n}u^4 - u^2v^2)\rangle \\
 &\quad + \langle(1, u, v, 0, u^2v - nv^3, -\frac{1}{n}u^3 + uv^2, -u^2v^2 + nv^4 + \frac{1}{n}u^4 - u^2v^2)\rangle \\
 &= \langle(2, 0, 0, 0, 0, 0, 2(n^2v^4 - 2u^2v^2 + \frac{u^4}{n}))\rangle \\
 &= 2\langle(1, 0, 0, 0, 0, \frac{(u^2 - nv^2)^2}{n})\rangle.
 \end{aligned}$$

Therefore we need to count the number of solutions to the equation $(u^2 - nv^2)^2 = nt$. Let θ be a root of the equation $X^2 = n$ and $E = F(\theta)$ be a quadratic extension field of $F = GF(q)$. We begin by proving the following lemma.

Lemma 4.7. *If d is a non-zero square in the field F , then there are $q + 1$ solutions $(u, v) \in F \times F$ to the equation $u^2 - nv^2 = d$.*

Proof. First note that $u^2 - nv^2$ is a square in the field F if and only if $u + \theta v$ is a square in the field E . The field E has $(q^2 - 1)/2$ non-zero squares and the field F has $(q - 1)/2$ non-zero squares. It follows that there are $(q^2 - 1)/2$ solutions $(u, v) \in F \times F$ to $u^2 - nv^2 = d$ where $d \neq 0$ is a square in F . Since there are $(q - 1)/2$ non-zero squares in F , there are $\frac{q^2-1}{2}/\frac{q-1}{2} = q + 1$ pairs of solutions to $u^2 - nv^2 = d$ for each non-zero square. ■

As a corollary to this, we have the following.

Corollary 4.8. *If e is a nonsquare in the field F , then there are $q + 1$ solutions $(u, v) \in F \times F$ to the equation $nv^2 - u^2 = e$.*

Proof. The solution set of the equation $nv^2 - u^2 = e$ is the same as the solution set of $n^2v^2 - nu^2 = ne$ and by Lemma 4.7, there are $q + 1$ solutions. ■

We now count the number of hyperbolic lines in the ovoidal graph that pass through the nonsingular point $\langle(1, 0, 0, 0, 0, 0, t)\rangle$ with the property that both u and v are non-zero.

Lemma 4.9. *If u and v are both non-zero, there are $q + 1$ hyperbolic lines satisfying equation (9) that pass through $\langle(1, 0, 0, 0, 0, 0, t)\rangle$ where t is a nonsquare.*

Proof. By the remarks following Lemma 4.6, we know that $nt = (u^2 - nv^2)^2$. Therefore we can write $nt = s^2$ where $s \in F$. If s is a square, then there are $q + 1$ solutions (u, v) to the equation by Lemma 4.7. If s is a nonsquare, then there are $q + 1$ solution to $nv^2 - u^2 = s$ by Corollary 4.8.

Depending upon whether -1 is a square or nonsquare in the field F , there will also be $q + 1$ solutions to either $u^2 - nv^2 = -s$ or $nv^2 - u^2 = s$. Thus there are $2(q + 1)$ solutions (u, v) to the equation $(u^2 - nv^2)^2 = s^2$. For each solution (u, v) there is a companion solution $(-u, -v)$ which determine a hyperbolic line in the ovoidal graph that passes through the nonsingular point $\langle (1, 0, 0, 0, 0, t) \rangle$. ■

Now consider the case when $u = v = 0$.

Lemma 4.10. *Let $P = \langle (1, 0, 0, 0, 0, t) \rangle$ and $u = v = 0$ in formula (9).*

- a. *If t is a square, there are $(q - 3)/2$ hyperbolic lines satisfying equation (9) that are incident with P .*
- b. *If t is a nonsquare, there are $(q - 1)/2$ hyperbolic lines satisfying equation (9) that are incident with P .*

Proof. Using the notation above, it follows that $x = y = 0$, $z = -aw$ and $(1 + a)t = -z^2 - aw^2 = -a^2w^2 - aw^2 = -(a^2 + a)w^2$. Therefore, $t = -aw^2$.

If t is a square, then $-a$ is a square, and w is a solution to the quadratic equation $X^2 = -t/a$. Since $a \neq -1$, there are $(q - 1)/2 - 1 = (q - 3)/2$ choices for w^2 . Thus there are $(q - 3)/2$ pairs of solutions $w_1, -w_1$ to the equations of the form $X^2 = -t/a$ and each pair determines a hyperbolic line incident with $\langle (1, 0, 0, 0, 0, t) \rangle$ where t is a square.

If t is a nonsquare, then for any square w^2 in the field, we can choose $a = -t/w^2$. Hence there are $(q - 1)/2$ choices for w^2 which leads to $(q - 1)/2$ hyperbolic lines in the ovoidal graph incident with $\langle (1, 0, 0, 0, 0, t) \rangle$ where t is a nonsquare. ■

We can now prove the following lemma which determines the number of hyperbolic lines of the ovoid that are incident with the line ℓ .

Lemma 4.11. *The number of hyperbolic lines of the Thas-Kantor ovoid that are incident with the nonsingular point P of the hyperbolic line ℓ satisfy the following.*

- a. *If P is a square nonsingular point, then P is incident with $(q - 1)/2$ hyperbolic lines of the ovoid.*
- b. *If P is a nonsquare point, then P is incident with $3(q + 1)/2$ hyperbolic lines of the ovoid.*

Proof. To prove (a), we note that by Lemma 4.10 there are $(q - 3)/2$ hyperbolic lines of the form given by equation (9) that are incident with P . Hence there are $(q - 3)/2 + 1 = (q - 1)/2$ hyperbolic lines in the ovoid incident with P .

To see (b), we have by Lemma 4.9 that there are $q + 1$ hyperbolic lines satisfying equation (9) if both u and v are nonzero. If $u = v = 0$, then there are $(q - 1)/2$ hyperbolic lines satisfying (9) by Lemma 4.10. Hence the number of hyperbolic lines of the ovoid passing through P is $q + 1 + (q - 1)/2 + 1 = 3(q + 1)/2$. ■

4.3. Proof of Theorem 4.1.

Proof. We consider the cases given in Theorem 4.1.

1. Let $\mathcal{O}$ be an $O_2^+(q)$ ovoid. Then $\mathcal{OG}$ is a star graph with $|\mathcal{HL}| = 1$. Hence the statement (1) can be checked easily.
2. Let $\mathcal{O}$ be an $O_3(q)$ ovoid. Note that $\mathcal{O}$ contains all the $q + 1$ singular points in $O_3(q)$. Let $\langle x \rangle$ be any nonsingular point. Then $\langle x \rangle^\perp = \{ \langle y \rangle \in PG(3, q) \mid f(x, y) = 0 \}$ is an $O_2^+(q)$ or $O_2^-(q)$ space when q is odd (depending on $\langle x \rangle$ is a square or nonsquare) and a degenerate line with one singular point if q is even. When q is odd, we may assume that we get $O_2^+(q)$ as $\langle x \rangle^\perp$ when $\langle x \rangle$ is a square. Note that any hyperbolic line incident with $\langle x \rangle$ does not intersect $\langle x \rangle^\perp$ in a singular point.

Assume q is even. Thus $\langle x \rangle$ is incident with $[(q + 1) - (1)]/2$ hyperbolic lines. Hence when $q > 2$, $\mathcal{S}$ is a partial linear space with parameters $s = q - 2$ and $t = (q - 2)/2$. Assume $q = 2$. It can be checked that the ovoidal graph $\mathcal{OG}$ of $O_3(2)$ is (δ, γ) -regular with $\delta = 1, \gamma = 1$.

Assume q is odd. If $\langle x \rangle$ is a square then $\langle x \rangle$ is incident with $[(q+1)-(2)]/2$ hyperbolic lines. Hence when $q > 3$, $\mathcal{S}_{sq}$ is a partial linear space with parameters $s_{sq} = (q-3)/2$ and $t = (q-3)/2$. If $\langle x \rangle$ is a nonsquare then $\langle x \rangle$ is incident with $[(q+1)-(0)]/2$ hyperbolic lines. Hence when $q > 3$, $\mathcal{S}_{nsq}$ is a partial linear space with parameters $s_{nsq} = (q-3)/2$ and $t = (q-1)/2$. Assume $q = 3$. It can be checked that the subgraphs of $\mathcal{OG}$ induced by $\mathcal{SQ}$ and $\mathcal{NSQ}$ of $O_3(3)$ are (δ, γ) -regular with $\delta = 1, \gamma = 1$.

Hence the statement (2) holds.

3. Let $\mathcal{O}$ be an $O_4^-(q)$ elliptic ovoid. Note that $\mathcal{O}$ contains all the singular points in $O_4^-(q)$. Let $\langle x \rangle$ be any nonsingular point. Then $\langle x \rangle^\perp = \{\langle y \rangle \in PG(3, q) | f(x, y) = 0\}$ is an $O_3(q)$ space. Note that any hyperbolic line incident with $\langle x \rangle$ does not intersect $\langle x \rangle^\perp$ in a singular point. Thus $\langle x \rangle$ is incident with $[(q^2+1)-(q+1)]/2$ hyperbolic lines. Hence when $q > 2$, $\mathcal{S}$ is a partial linear space with parameters $s = q-2$ and $t+1 = (q^2-q)/2$. Assume $q = 2$. It can be checked that the ovoidal graph $\mathcal{OG}$ of $O_4^-(2)$ is (δ, γ) -regular with $\delta = 1, \gamma = 1$.

Assume $q(> 3)$ is odd. Since each hyperbolic line contains $(q-1)/2$ square points and $(q-1)/2$ nonsquare points, $\mathcal{S}_{sq}$ and $\mathcal{S}_{nsq}$ are partial linear spaces with parameters $s_{sq} = s_{nsq} = (q-3)/2$ and $t_{sq} = t_{nsq} = t = (q^2-q-2)/2$. Assume $q = 3$. It can be checked that the subgraphs of $\mathcal{OG}$ induced by $\mathcal{SQ}$ and $\mathcal{NSQ}$ of $O_4^-(3)$ are (δ, γ) -regular with $\delta = 1, \gamma = 1$. Hence the statement (3) holds.

4. Let $\mathcal{O}$ be the Tits ovoid in the space $O_5(q)$ where $q = 2^{2h+1}, h > 0$. By Lemma 4.5, every nonsingular point in the ovoidal graph is incident with $q/2$ hyperbolic lines of the ovoid. Thus $\mathcal{S}$ is a partial linear space with $s = q-2$, and $t = (q-2)/2$. Hence the statement (4) holds.
5. Let $\mathcal{O}$ be an ovoid in $O_7(3)$. The Kantor's construction given in Section 4.2, with $h = 1$ produces $O_7(3)$ ovoid. The formulas in Lemma 4.11 are still valid. Hence the subgraph of $\mathcal{OG}$ induced by $\mathcal{SQ}$ is (δ, γ) -regular with $\delta = 1, \gamma = 1$, and the subgraph of $\mathcal{OG}$ induced by $\mathcal{NSQ}$ is (δ, γ) -regular with $\delta = 1, \gamma = 6$.
6. Let $q = 3^h$ where $h \geq 2$ and $\mathcal{O}$ be the Thas-Kantor ovoid in the parabolic space $O_7(q)$. By Lemma 4.11, there are $(q-1)/2$ hyperbolic lines that are incident with any square point, and $3(q+1)/2$ hyperbolic lines that are incident with any nonsquare point. Thus $\mathcal{S}_{sq}$ is a partial linear space with $s_{sq} = (q-3)/2$, $t_{sq} = (q-3)/2$, and $\mathcal{S}_{nsq}$ is a partial linear space with $s_{nsq} = (q-3)/2$, and $t_{nsq} = (3q+1)/2$. Hence the statement (6) holds.
7. Let $\mathcal{O}$ be an ovoid in $O_8^+(2)$. This ovoid can be produced using the construction given in the proof of (8). It can be checked that the ovoidal graph $\mathcal{OG}$ is (δ, γ) -regular with $\delta = 1, \gamma = 1$.
8. Let $\mathcal{O}$ be a Kantor $PSL(2, q^3)$ ovoid where $q = 2^h, h \geq 2$. In [19], Kantor describes the ovoid $\mathcal{O}$ as follows. Let $K = GF(q)$ and $F = GF(q^3)$, and $V = K \oplus F \oplus K$ equipped with a quadratic form Q , defined by $Q(a, \beta, \alpha, d) = ad + Tr(\beta\alpha)$, where $Tr(\beta\alpha) = (\beta\alpha)^q + (\beta\alpha)^{q^2}$. Then $\mathcal{O} = \{ \langle (0, 0, 0, 1) \rangle \} \cup \{ \langle (1, x, x^{q+q^2}, N(x)) \rangle : x \in F \}$, where $N(x) = x^{1+q+q^2}$. The group $G = PSL(2, q^3)$ of orthogonal transformations acts 3-transitively on $\mathcal{O}$.

Let $u = (0, 0, 0, 1)$, $v = (1, 0, 0, 0)$, and $w = (1, 1, 1, 1)$. It is easy to check that $\langle u \rangle, \langle v \rangle, \langle w \rangle \in \mathcal{O}$. Any non singular point on $\langle u, v \rangle \in \mathcal{L}$ can be represented by $\langle (1, 0, 0, y) \rangle$, where $y \in GF(q) \setminus \{0\}$. Now consider the line $\ell = \langle w, (1, 0, 0, y) \rangle$. The line ℓ is nondegenerate when $y \neq 1$. When $y \neq 1$, consider any point $g \notin \langle w \rangle^\perp$ on ℓ . So $g = \langle (1 + \alpha, 1, 1, 1 + \alpha y) \rangle$ for some $\alpha \in GF(q) \setminus \{0, 1\}$. Suppose $g \in \mathcal{O}$. Then $(1 + \alpha)t = 1$ and $(1 + \alpha)t^{q+q^2} = 1$. Since $\alpha \neq 1$, we have $t^{q^2+q-1} = 1$ and hence q^2+q-1 divides q^3-1 contradicting $q > 1$. Hence any hyperbolic line through $\langle w \rangle$ that intersects ℓ is tangent to $\mathcal{O}$. Since $\mathcal{O}$ is a 3-transitive ovoid, $\mathcal{L}$ contains a set of parallel lines. Thus $\mathcal{S}$ is an incidence structure with parameters $s = (q-2)$ and $t = 0$. Hence the statement (8) holds.

Now we apply Lemma 4.2 to the above mentioned cases to get the ovoidal graph invariants shown in Table 2. This completes the proof of Theorem 4.1. ■

5. OVOIDAL GRAPH INVARIANTS

In this section, we present our ovoidal graph invariants for a selective set of known ovoids representing various orthogonal spaces. Note that ovoids do not exist in $O_{2m}^-(q)$ ($m \geq 3$) [33], $O_{2m}^+(q)$ ($q = 2^e, 3^e, m \geq 5$ and $q = 5^e, 7^e, m \geq 6$) [4, 29], $O_{2m-1}(q)$ ($m \geq 5$), [18, 4] and $O_7(q)$ ($q \neq 3$ and q is a prime) [25, 2]. No construction is known for the remaining spaces of $O_{2m}^+(q)$ ($m \geq 5$) and $O_8^+(q)$ ($q \equiv 1 \pmod 6$ and q is not a prime).

In Table 2, we give the ovoidal graph invariants for the 2-transitive ovoids of orthogonal spaces mentioned in Theorem 4.1. Omitted from this table are the 2-transitive ovoids in $O_4^+(q)$ stabilized by $\text{Aut}(PSL_2(q))$, the Ree-Tits ovoids in $O_7(q)$ stabilized by $\text{Aut}({}^2G_2(q))$, $q = 3^{2h+1} \geq 27$, and the Kantor's ovoids in $O_8^+(q)$ stabilized by $\text{Aut}(PSU_3(q))$, $q \geq 5$, $q \equiv 2 \pmod 3$.

Computational results show that our invariant is complete for all known ovoids in: $O_5(q)$ where $q \leq 3^3$, $O_6^+(q)$ for $q \leq 7$, $O_8^+(q)$ for $q \leq 5$. For many cases, it is enough to use just $|\mathcal{P}|$ or $\{|\mathcal{SQ}|, |\mathcal{NSQ}|\}$ to identify ovoids.

TABLE 3. Ovoidal Graph Invariants for some $O_5(q)$ ovoids

Space	Ovoid	$ \mathcal{P} $	$ \mathcal{SQ} $	$ \mathcal{NSQ} $
$O_5(9)$	Kantor	6141	3069	3072
$O_5(25)$	Kantor	378,655	189,325	189,330
$O_5(27)$	Kantor	521,262	260,469	260,793
$O_5(27)$	Ree-Tits Slice	539,010	264,654	265,356
$O_5(27)$	Thas-Payne	528,371	260,469	259,902

It is known that, when q is a prime, the elliptic quadrics ($O_4^-(q)$) are the only $O_5(q)$ ovoids [2]. The known families of ovoids in $O_5(q)$ are the Kantor ovoids [19], the Ree-Tits slice ovoids $q = 3^{2h+1}$, $h > 0$ [19], the Thas-Payne ovoids $q = 3^h$, $h > 2$ [34], and the Tits ovoids $q = 2^{2h+1}$, $h > 0$ [32]. For $q = 9$ ovoids are either elliptic quadrics or Kantor [14]. For $q = 4, 16$ only elliptic quadrics occur and $q = 8, 32$ just elliptic quadric and Tits ovoids arise [25, 35]. For $q = 25$ the only known ovoids are elliptic quadrics and Kantor. For $q = 27$ the only known ovoids are elliptic quadrics, the Kantor ovoid, the Ree-Tits Slice ovoid, and the Thas-Payne ovoid. In Table 3, we show the isomorphism invariant for all the known ovoids that are not 2-transitive in the parabolic spaces $O_5(q)$ where $q \leq 27$. Note that the isomorphism invariant $|\mathcal{P}|$ is sufficient to identify the ovoids in $O_5(q)$ where $q = 9, 25$ or 27.

TABLE 4. Ovoidal Graph Invariants for the $O_6^+(5)$ ovoids

Ovoid	$ \mathcal{P} $	$ \mathcal{SQ} $	$ \mathcal{NSQ} $	D_{sq}	D_{nsq}
$\mathcal{O}_1$	908	478	430	$\{(1, 440), (5, 33), (9, 5)\}$	$\{(1, 290), (2, 120), (6, 20)\}$
$\mathcal{O}_2$	1006	446	560	$\{(1, 340), (2, 45), (3, 30), (4, 30), (10, 1)\}$	$\{(1, 545), (7, 15)\}$
$\mathcal{O}_3$	574	314	260	$\{(1, 240), (3, 9), (5, 24), (6, 30), (7, 10), (13, 1)\}$	$\{(1, 120), (2, 75), (4, 20), (6, 30), (8, 15)\}$
$\mathcal{O}_4$	853	443	410	$\{(1, 306), (2, 102), (3, 24), (4, 5), (8, 6)\}$	$\{(1, 260), (2, 102), (3, 24), (4, 9), (5, 12), (6, 3)\}$
$\mathcal{O}_5$	808	404	404	$\{(1, 208), (2, 186), (4, 1), (6, 3), (8, 6)\}$	$\{(1, 260), (2, 72), (3, 66), (7, 3), (9, 3)\}$
$\mathcal{O}_6$	765	405	360	$\{(1, 200), (2, 200), (10, 5)\}$	$\{(1, 150), (2, 150), (3, 50), (5, 10)\}$
$\mathcal{O}_7$	1014	507	507	$\{(1, 377), (2, 117), (3, 13)\}$	$\{(1, 377), (2, 117), (3, 13)\}$
$\mathcal{O}_8$	964	486	478	$\{(1, 380), (2, 78), (3, 16), (5, 6), (6, 6)\}$	$\{(1, 387), (2, 48), (3, 8), (4, 32), (5, 3)\}$
$\mathcal{O}_9$	988	506	482	$\{(1, 408), (2, 60), (3, 36), (7, 2)\}$	$\{(1, 338), (2, 132), (4, 12)\}$
$\mathcal{O}_{10}$	800	410	390	$\{(1, 210), (2, 190), (3, 4), (8, 6)\}$	$\{(1, 160), (2, 225), (8, 5)\}$
$\mathcal{O}_{11}$	628	278	350	$\{(1, 50), (2, 168), (3, 24), (5, 24), (6, 12)\}$	$\{(1, 241), (3, 96), (9, 12), (13, 1)\}$
$\mathcal{O}_{12}$	970	485	485	$\{(1, 395), (2, 27), (3, 60), (7, 3)\}$	$\{(1, 395), (2, 27), (3, 60), (7, 3)\}$
$\mathcal{O}_{13}$	928	464	464	$\{(1, 348), (2, 84), (3, 12), (4, 11), (6, 9)\}$	$\{(1, 358), (2, 66), (3, 9), (4, 24), (5, 6), (7, 1)\}$
$\mathcal{O}_{14}$	957	472	485	$\{(1, 334), (2, 114), (3, 12), (4, 10), (6, 2)\}$	$\{(1, 356), (2, 105), (3, 21), (7, 3)\}$
$\mathcal{O}_{15}$	898	437	461	$\{(1, 284), (2, 108), (3, 42), (8, 3)\}$	$\{(1, 366), (2, 36), (3, 45), (4, 1), (5, 9), (6, 3), (10, 1)\}$
$\mathcal{O}_{16}$	964	482	482	$\{(1, 354), (2, 104), (3, 13), (4, 8), (5, 1), (6, 2)\}$	$\{(1, 354), (2, 104), (3, 13), (4, 8), (5, 1), (6, 2)\}$
$\mathcal{O}_{17}$	796	398	398	$\{(1, 265), (2, 54), (3, 48), (4, 24), (5, 6), (7, 1)\}$	$\{(1, 265), (2, 54), (3, 48), (4, 24), (5, 6), (7, 1)\}$
$\mathcal{O}_{18}$	688	344	344	$\{(1, 144), (2, 144), (3, 24), (4, 27), (6, 3), (10, 2)\}$	$\{(1, 144), (2, 144), (3, 24), (4, 27), (6, 3), (10, 2)\}$
$\mathcal{O}_{19}$	1122	561	561	$\{(1, 486), (2, 66), (3, 4), (4, 5)\}$	$\{(1, 486), (2, 66), (3, 4), (4, 5)\}$
$\mathcal{O}_{20}$	130	65	65	$\{(10, 65)\}$	$\{(10, 65)\}$

Charnes [6] has determined that there are 20 non-isomorphic ovoids in $O_6^+(5)$. The ovoidal graph invariants differentiate these ovoids, however in this case there are two non-isomorphic ovoids with the

same $|\mathcal{P}|$. In this case the invariants $|\mathcal{SQ}|$ and $|\mathcal{NSQ}|$ are needed to differentiate all the ovoids. In Table 4, the ovoid O_i corresponds to the ovoid with fingerprint Ξ_i in [6, Table 1, p.159].

In $O_6^+(7)$, there are 1347 translation planes of which 374 are polar pairs. Hence there are 973 non-isomorphic ovoids $O_6^+(7)$ [21]. Gordon Royle provides a complete list of the 1347 translation planes on his web page [28]; Charnes and Dempwolff [7] calculate the automorphism groups and fingerprints of these ovoids. In this orthogonal space, the ovoidal graph invariants $(|\mathcal{SQ}|, |\mathcal{NSQ}|)$ separate 959 of the ovoids while D_{sq}, D_{nsq} separates all the 973 non-isomorphic ovoids.

It is known that an $O_8^+(q)$ space has a unique ovoid when $q = 2, 3$ [19], or $q = 4$ [16] which is 2-transitive and mentioned in Table 2. According to [8], the only $O_8^+(5)$ ovoids are the Kantor ovoid (2-transitive) [19], the Cooperstein ovoid [10] which is the only primitive ovoid that is not 2-transitive [17, 3], and the binary ovoid constructed from the $E8$ root lattice [9, 23]. In Table 5, we show the invariants for the $O_8^+(5)$ ovoids and the Dye ovoid [12] in $O_8^+(8)$ which is the only known sporadic ovoid in $O_8^+(q)$.

TABLE 5. Ovoidal Graph Invariants for some $O_8^+(q)$ ovoids

Space	Ovoid	$ \mathcal{P} $	$ \mathcal{SQ} $	$ \mathcal{NSQ} $	D_{sq}	D_{nsq}
$O_8^+(5)$	Kantor	13125	7875	5250	$\{(2, 7875)\}$	$\{(3, 5250)\}$
$O_8^+(5)$	Cooperstein	20100	14220	5880	$\{(1, 14175), (35, 45)\}$	$\{(1, 3150), (3, 2100), (10, 630)\}$
$O_8^+(5)$	Binary $E8$ Root Lattice	13924	9640	4284	$\{(1, 7560), (3, 2016), (33, 63), (63, 1)\}$	$\{(2, 3024), (5, 756), (10, 378), (17, 126)\}$
$O_8^+(8)$	Dye	306936	0	306936	$\{(2, 226800), (4, 56700), (6, 15120), (12, 2268), (16, 1512), (18, 3024), (22, 720), (28, 756), (148, 36)\}$	

We have also found examples of orthogonal spaces in which our invariant is not complete. The ovoidal graph invariants are the same for the two ovoids in $O_4^+(32)$ mentioned in Table 1. Also the Ree-Tits and Thas-Kantor ovoids over the field $GF(27)$ give the same ovoidal graph invariants. Another interesting example is $O_4^+(16)$ space which contains two isomorphism classes of 2-transitive ovoids with the invariants $D_{sq} = \{1, 2040\}$ and $D_{sq} = \{(1, 1632), (6, 68)\}$. The latter ovoid provides an example of a 2-transitive ovoid which produces an ovoidal graph that is not (δ, γ) -regular.

It will be interesting to find the ovoidal graph invariants for the remaining 2-transitive ovoids, namely, the ovoids in $O_4^+(q)$ stabilized by $Aut(PSL_2(q))$, the Ree-Tits ovoids in $O_7(q)$ stabilized by $Aut(^2G_2(q))$, $q = 3^{2h+1} \geq 27$, and the Kantor's ovoids in $O_8^+(q)$ stabilized by $Aut(PSU_3(q))$, $q \geq 5$, $q \equiv 2 \pmod{3}$. It can be observed that some of the graphs in Table 2 contain regular bipartite subgraphs of degree $(q-1)/2$ (e.g., $O_3(q)$ (q odd), Thas-Kantor $O_7(3^h)$).

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